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Abstract

Extended Chebyshev spaces are natural generalisations of polynomial spaces due to the same upper bounds on the number of zeroes. In a natural approach, many results of the polynomial framework have been generalised to the larger Chebyshevian framework, concerning Approximation Theory as well as Geometric Design. In the present work, we go the reverse way: considering polynomial spaces as examples of Extended Chebyshev spaces, we apply to them results specifically developed in the Chebyshevian framework. On a closed bounded interval, each Extended Chebyshev space can be defined by means of sequences of generalised derivatives which play the same rôle as the ordinary derivatives for polynomials. We recently achieved an exhaustive description of the infinitely many such sequences. Surprisingly, this issue is closely related to the question of building positive linear operators of the Bernstein type. As Extended Chebyshev spaces, one can thus search for all generalised derivatives which can be associated with polynomials spaces on closed bounded intervals. Though this may a priori seem somewhat nonsensical due to the simplicity of the ordinary derivatives, this actually leads to new interesting results on polynomial and rational Bernstein operators and related results of convergence.

Keywords: Bernstein bases, Bernstein operators, Extended Chebyshev spaces, generalised derivatives, weight functions, blossoms, polynomial and rational spaces.

MSC: Primary 41A36; Secondary 41120, 41A50, 65D17.

§1. Introduction

In the introduction I will denote a fixed given non-trivial real interval, and $\mathbb{P}_n$ the restriction to I of the polynomial space of degree n. Moreover, for the sake of simplicity, all functions we consider will be assumed to be infinitely many times differentiable on I.

On the interval I, the class $\mathcal{C}$ of all Extended Chebyshev spaces of any dimension is the natural generalisation of the class $\mathcal{P}$ of all polynomial spaces $\mathbb{P}_n$, $n \geq 0$, with which they share the same upper bounds on the number of zeroes of non-zero elements. In particular, though not as simple to handle, the class $\mathcal{C}$ can serve as a useful substitute for the class $\mathcal{P}$ to obtain unique solutions to Hermite interpolation problems. Indeed, it presents real advantages compared to $\mathcal{P}$, due both to its great variety and to the shape parameter(s) each Extended Chebyshev space offers. In front of such a generalised situation, the logical approach for a mathematician consists in taking each single known result of the class $\mathcal{P}$, and try to extend it to the larger class $\mathcal{C}$. However, this turns out to be not so easy, for two underlying facts are crucial in many ways in the polynomial framework. Firstly, we have a nested sequence

$$\mathbb{P}_0 \subset \mathbb{P}_1 \subset \mathbb{P}_2 \subset \cdots \subset \mathbb{P}_n. \tag{1.1}$$

Secondly, the class $\mathcal{P}$ is closed under the ordinary derivative D since

$$D\mathbb{P}_n := \mathbb{P}_{n-1}, \quad n \geq 1. \tag{1.2}$$

To quote only a most classical example, given any $x_0, \ldots, x_n \in I$, the n-th order divided difference based on $x_0, \ldots, x_n$ of any function F on I is defined as the leading coefficient of the unique polynomial $P_n \in \mathbb{P}_n$ which interpolates F at $x_0, \ldots, x_n$, in the Hermite sense. The famous recurrence relations for divided differences readily follows from (1.1), along with the so important Newton expression of the interpolating polynomial P_n. No similar possibility naturally appears in the class $\mathcal{C}$, unless we more closely imitate the class $\mathcal{P}$ by restricting ourselves to nested sequences

$$\mathbb{E}_0 \subset \mathbb{E}_1 \subset \mathbb{E}_2 \subset \cdots \subset \mathbb{E}_n, \tag{1.3}$$

where each $\mathbb{E}_i$ is a $(i+1)$-dimensional Extended Chebyshev space on I. Indeed, each such sequence (1.3) naturally gives sense to the concept of *leading coefficient* in each $\mathbb{E}_i$ by selecting once and for all a function $U_i \in \mathbb{E}_i \setminus \mathbb{E}_{i-1}$ for each $i \geq 0$, with $\mathbb{E}_{i-1} := \{0\}$. This in turn naturally leads to the concept of associated *generalised divided differences*, associated recurrence relations and Newton-type formula for the unique element $F_n \in \mathbb{E}_n$ which interpolates F at $x_0, \ldots, x_n$. Concerning this specific issue, definitions and proofs can easily be adapted from the polynomial framework to all Chebyshevian situations (1.3). This corresponds to the subclass $\mathcal{C}_0 \subset \mathcal{C}$ formed by the so-called *Extended Complete Chebyshev spaces on I*.

What about the case where $x_0 = \ldots = x_n = \xi$, corresponding to Taylor interpolation of order n of F at ξ? As is well known, the corresponding divided difference is equal to $F^{(n)}(\xi)/(n!)$. This obviously fails to be true in general in the class $\mathcal{C}_0$. What is thus the corresponding result? This is actually strongly connected with the question: what is the analogue of (1.2) within the class $\mathcal{C}$? Well, we precisely face there a major difference between the polynomial and the Chebyshevian frameworks: unlike $\mathcal{P}$, the class $\mathcal{C}$ is not closed under D. Applying D to an Extended Chebyshev space decreases the dimension only when it contains constants, but even so, there is absolutely no guarantee that the resulting space will be an Extended Chebyshev space on I.

To overcome this difficulty, one can try to go still closer to the class $\mathcal{P}$ by considering *generalised monomials* associated with any given sequence $(w_0, \dots, w_n)$ of positive functions on I, defined on I by $U_0(t) := w_0(t)$ and

$$U_i(t) := w_0(t) \int_a^t w_1(\xi_1) \int_a^{\xi_1} w_2(\xi_2) \dots \int_a^{\xi_{i-1}} w_i(\xi_i) d\xi_i \dots d\xi_1, \quad i = 1, \dots, n, \tag{1.4}$$

where a is any point in I. Each space $\mathbb{E}_i := \mathrm{span}(U_0, \dots, U_i)$, $0 \le i \le n$, can be proved to be an $(i+1)$-dimensional Extended Chebyshev space on I. Accordingly, these spaces all belong to the class $\mathcal{C}_0$. Now, if L_0 stands for the division by w_0, the operator DL_0 transforms the corresponding nested sequence (1.3) into the shorter nested sequence $DL_0\mathbb{E}_i$, $1 \le i \le n$, built exactly the same way, but now from the generalised monomials associated with $(w_1, \dots, w_n)$, instead of $(w_0, w_1, \dots, w_n)$. Iterating the process, we obtain a sequence DL_i, $i = 0, \dots, n$, of differential operators which is to the nested sequence $\mathbb{E}_i := \mathrm{span}(U_0, \dots, U_i)$, $i = 0, \dots, n$, the analogue of what the sequence D^i, $i \le n+1$, is to the sequence (1.1). Note that the corresponding generalised divided difference based on $x_0, \dots, x_n$ when they all are equal to some ξ is equal to $L_nF(\xi)$. We thus have the answers to our questions within the subclass $\mathcal{C}_1$ of $\mathcal{C}_0$ composed of all Extended Chebyshev spaces spanned by generalised monomials. The good news is that both classes $\mathcal{C}_1$ of $\mathcal{C}_0$ coincide. Classically, all approximation properties of Extended Chebyshev spaces, *e.g.*, Taylor expansions, generalised convexity, ..., were developed within the class $\mathcal{C}_0$, by means of generalised monomials [11, 24, 30, 14].

Now, what about our initial class $\mathcal{C}$? Well, it does coincide too with the class $\mathcal{C}_0$, but only if the interval I is a closed and bounded interval $[a, b]$. Let us now work under this requirement, which is not a real limitation for most issues. Within $\mathcal{C}$, we are thus now in the situation (1.3), which seems the exact counterpart of (1.1). However, in spite of the similarities, in practice a huge difference still exists between $\mathcal{C}$ and $\mathcal{P}$. Life is quiet in the polynomial class $\mathcal{P}$ because, in the space $\mathbb{P}_n$, the nested sequence (1.1) is well identified, just as the sequence of ordinary derivatives D^i to decrease the dimension down to zero. These useful tools are so simple that nobody would reasonably think of searching for replacements. Life is much less quiet in the larger class $\mathcal{C}$. Indeed, with a given $(n+1)$-dimensional space $\mathbb{E}_n$ of this class one can associate infinitely many nested sequences (1.3). In general none of them seems preferable to the others. The problem is that a result proved using the generalised derivatives associated with one of these nested sequences may not be intrinsic. This motivated the search of all possible such nested sequences, that is, all possible generalised derivatives which can be associated with $\mathbb{E}_n$. The result was achieved in [19]. Surprisingly, this issue is very closely connected to the construction of all linear operators of the Bernstein-type based on $\mathbb{E}_n$. This is the reason why each $\mathbb{E}_n$ of the class $\mathcal{C}$, of dimension at least three, provides us with infinitely many (Chebyshevian) Bernstein operators. In general, none of them seems preferable to the others. This may be seen as an inconvenience compared to the one and only positive linear operator allocated to the space $\mathbb{P}_n$ after S. Bernstein [2]. We do prefer to interpret this as a real advantage and a great richness of the Chebyshevian world $\mathcal{C}$.

Returning to the polynomial class $\mathcal{P}$, the object of the present article is to give it the benefit of the results achieved is the larger Chebyshevian class $\mathcal{C}$ via difficult tools and techniques which are a priori unjustified in $\mathcal{P}$. It is worthwhile mentioning that it is not the first time we develop a similar approach. Indeed, the knowledge of all systems of generalised derivatives in each polynomial space $\mathbb{P}_n$ already proved to be fruitful for CAGD purposes in [20]. We explained there how to use them to deduce necessary and sufficient conditions on the connection matrices for the associated space of geometrically continuous polynomial splines to be suitable for design. Here we present some implications concerning either polynomial or rational Bernstein operators in the Chebyshevian sense, with corresponding results of convergence. This is done is Sections 4 and 5, respectively. Beforehand, Section 2 gathers some technical preliminary results related to the convergence of the classical polynomial Bernstein operators which will be very useful to achieve convergence results in Sections 4 and 5. Section 3 briefly surveys the question of generalised derivatives and Bernstein operators based on Extended Chebyshev spaces. Section 6 presents a few final remarks for future related work. It is worthwhile mentioning that an interesting class of rational Bernstein operators was introduced by P. Piţul and P. Sablonnière in [26]. They are actually Bernstein operators in the sense of Extended Chebyshev spaces. Some of our results are closely related to them and a detailed comparison is carried out in Section 5.

§2. A brief reminder about polynomials

In this section we fix some notations and we mainly remind readers with known results and related proofs, all connected with the famous Bernstein operators. They will strongly be involved in Sections 4 and 5.

Throughout the rest of the article, for any nonnegative integer k, $\mathbb{P}_k$ denotes the degree k polynomial space restricted to $[0,1]$, and $(B_0^k, \dots, B_k^k)$ the polynomial Bernstein basis

$$B_i^k(t) := \binom{k}{i} t^i (1-t)^{k-i}, \quad t \in [0,1], \quad i = 0, \dots, k. \tag{2.1}$$

We also denote by $\mathbb{B}_k^* : C^0([0,1]) \to \mathbb{P}_k$ the classical Bernstein operator of degree k [2, 4, 13, 5]. As is well-known, it is defined by

$$\mathbb{B}_k^* F := \sum_{i=0}^{k} F(t_{k,i}^*) B_i^k \quad \text{for all } F \in C^0([0,1]), \quad \text{with } t_{k,i}^* := \frac{i}{k}, \quad i = 0, \dots, k. \tag{2.2}$$

Given an integer $n \geq 1$, let X_n denote the monomial function $X_n(t) := t^n$, for $t \in [0,1]$. For any $k \geq n$, we can consider the Bézier points of X_n, in the sense of the coefficients of X_n when expanded in terms of the Brenstein polynomial basis of degree k. Let us denote them by $x_{n,k,i}$, $0 \leq i \leq k$, so that

$$X_1 = \sum_{i=0}^{k} x_{1,k,i} B_i^k = \sum_{i=0}^{k} t_{k,i}^* B_i^k \text{ for any } k \geq 1, \quad X_n = \sum_{i=0}^{k} x_{n,k,i} B_i^k, \text{ for any } k \geq n \geq 2.$$

Any polynomial Bernstein basis being *normalised*, that is, $\sum_{i=0}^{k} B_i^k = \mathbb{I}$, (where $\mathbb{I}$ denotes the constant function $\mathbb{I}(t) = 1$ for all $t \in [0,1]$), the left equality above is the reason why any Bernstein operator of degree $k \geq 1$ reproduces $\mathbb{P}_1$, in the sense that $\mathbb{B}_k F = F$ for all $F \in \mathbb{P}_1$.

We will see in Theorem 2.2 that Lemma 2.1 below permits an easy proof of the well-known convergence of the sequence $\mathbb{B}_k^*$, $k \geq 1$, via Korovkin's theorem. However this is not the reason why we need to cite it here. We actually need it for Proposition 2.3 which will be strongly involved in the proofs of the convergence results we will establish in Sections 4 and 5.

le:Xn **Lemma 2.1.** *For any $n \geq 2$, we have*

$$\lim_{k \to +\infty} \max_{0 \leq i \leq k} \left| X_n(t_{k,i}^*) - x_{n,k,i} \right| = 0. \tag{2.3}$$

Proof. Let us select any $k \geq n$. Considered as a polynomial of degree at most k, the monomial X_n possesses a blossom in k variables [28]. Let us denote it as $x_{n,k}$. It is the unique function of k variables which is symmetric, of degree at most one in each variable, and which gives X_n on the diagonal. It is therefore given by

$$x_{n,k}(t_1, \dots, t_k) = \frac{1}{\binom{k}{n}} \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq k} t_{i_1} t_{i_2} \dots t_{i_n}.$$

With the notation $x^{[j]}$ for x repeated j times, we know that

$$x_{n,k,i} := x_{n,k}\left(0^{[k-i]}, 1^{[i]}\right), \quad 0 \leq i \leq k.$$

Blossoms thus provide an easy way to obtain the known formula $x_{n,k,i} = \frac{\binom{i}{n}}{\binom{k}{n}}$, that is,

$$x_{n,k,i} = 0 \quad \text{for } 0 \leq i \leq n-1, \quad x_{n,k,i} = \frac{i(i-1)\dots(i-n+1)}{k(k-1)\dots(k-n+1)} \quad \text{for } n \leq i \leq k.$$

We thus have to check that

$$\lim_{k\to+\infty} \max_{n\leq i\leq k} \left| \left(\frac{i}{k}\right)^n - \frac{i(i-1)\dots(i-n+1)}{k(k-1)\dots(k-n+1)} \right| = 0. \tag{2.4}$$

For $n = 1$ there is nothing to prove. Besides, for $i \geq n+1$, we have

$$\left| \left(\frac{i}{k}\right)^{n+1} - \frac{i(i-1)\dots(i-n)}{k(k-1)\dots(k-n)} \right| \leq \frac{i}{k} \left| \left(\frac{i}{k}\right)^n - \left(\frac{i-1}{k-1}\right)^n \right| + \frac{i}{k} \left| \left(\frac{i-1}{k-1}\right)^n - \frac{(i-1)\dots(i-n)}{(k-1)\dots(k-n)} \right|.$$

Given that

$$\left| \left(\frac{i}{k}\right)^n - \left(\frac{i-1}{k-1}\right)^n \right| \leq 2^{n-1} \frac{(k-i)i}{k(k-1)},$$

the proof of (2.4) readily follows by induction on n. ∎

`th:bernstein` **Theorem 2.2** ([2]). *For any $V \in C^0([0,1])$, $\lim_{k\to+\infty} \|V - \mathbb{B}_k^* V\|_\infty = 0$.*

Proof. Since each positive linear operator $\mathbb{B}_n$ reproduces $\mathbb{1}, X_1$, Korovkin's theorem [12, 4, 5] says that it is sufficient to prove the claimed result for $F = X_2$. Now,

$$\|\mathbb{B}_k X_2 - X_2\|_\infty = \left\| \sum_{i=0}^{k} \left[X_2(t^*_{k,i}) - x_{2,k,i}\right] B_i^k \right\|_\infty \leq \max_{0\leq i\leq k} \left|X_2(t^*_{k,i}) - x_{2,k,i}\right|.$$

Lemma 2.1 guarantees that $\lim_{k\to+\infty} \|\mathbb{B}_k X_2 - X_2\|_\infty = 0$. ∎

As mentioned earlier the following consequence of Lemma 2.1 will be essential in the subsequent sections.

`prop:limitP` **Proposition 2.3.** *Let $P \in \mathbb{P}_n$ for some $n \geq 1$. Given any $k \geq n$, let $p_{k,i}$, $0 \leq i \leq k$, be the Bézier points of P considered as a polynomial of degree at most k. Then,*

$$\lim_{k\to+\infty} \max_{0\leq i\leq k} \left|P(t^*_{k,i}) - p_{k,i}\right| = 0 \tag{2.5}$$

Proof. Let us write P as $P = \sum_{\ell=0}^{n} A_\ell X_\ell$, with $X_0 = \mathbb{1}$. Consider any $k \geq n$. We have both

$$P(t^*_{k,i}) = \sum_{\ell=0}^{n} A_\ell X_\ell(t^*_{k,i}), \quad p_{k,i} = \sum_{\ell=0}^{n} A_\ell x_{\ell,k,i} \quad \text{for } i = 0, \dots, k, \tag{2.6}$$

with $x_{0,k,i} = 1$ for $i = 0, \dots, k$. Setting $\mathcal{A} := \max\left(|A_2|, \dots, |A_n|\right)$, it readily follows that

$$\max_{0 \leq i \leq k} \left|P(t^*_{k,i}) - p_{k,i}\right| \leq \mathcal{A} \sum_{\ell=2}^{n} \max_{0 \leq i \leq k} \left|X_\ell(t^*_{k,i}) - x_{\ell,k,i}\right|. \tag{2.7}$$

Accordingly (2.5) follows from Lemma 2.1. ■

ee elevation

Remark 2.4. Observe that Lemma 2.1 also permits an efficient proof of the convergence of degree elevation: with the notations of Proposition 2.3, for each $P \in \mathbb{P}_n$, the sequence $L_k P$, $k \geq n$, defined by

$$L_k P(t^*_{k,i}) = p_{k,i} \text{ for } i = 0, \dots, k, \quad L_k P \text{ is affine on } \left[t^*_{k,i}, t^*_{k,i+1}\right] \text{ for } i = 0, \dots, k-1,$$

uniformly converges to P on $[0, 1]$. Indeed, out of linearity, it suffices to prove this for any monomial X_n, $n \geq 2$. Simple convexity arguments show that

$$\mathbb{B}_k X_n \geq L_k\left(\mathbb{B}_k X_n\right) \geq X_n \geq L_k X_n, \quad k \geq n.$$

It follows that

$$\|X_n - L_k X_n\|_\infty \leq \|L_k(B_k X_n) - L_k X_n\|_\infty = \max_{0 \leq i \leq k} \left|X_n(t^*_{k,i}) - x_{n,k,i}\right|, \quad k \geq n.$$

Lemma 2.1 proves that $\lim_{k \to +\infty} \|X_n - L_k X_n\|_\infty = 0$.

§3. The Chebyshevian framework: generalised derivatives vs Bernstein operators

In the present section, we briefly present Extended Chebyshev spaces and the few main tools which this work relies on: Bernstein bases, Bernstein operators, weight functions and associated generalised derivatives, along with how they are connected together. For further acquaintance with this question readers are referred to [18, 19] and also [1], and more generally for further acquaintance with Extended Chebyshev spaces, to [11, 30, 14, 27, 15, 16, 17].

3.1. Extended Chebyshev spaces

Throughout the present section I stands for any non-trivial real interval, and $\mathbb{E} \subset C^n(I)$ is an $(n+1)$-dimensional linear space. Such a space is said to be an *Extended Chebyshev space on I* (for short, EC-space on I) if any non-zero $F \in \mathbb{E}$ vanishes at most n times on I, multiplicities included up to $(n+1)$, or, equivalently, if $\mathbb{E}$ permits Hermite interpolation on I, that is, if any Hermite interpolation problem in $(n+1)$ data in I has a unique solution in $\mathbb{E}$.

Example 3.1. Let $\mathbb{E}$ be the null space of any linear differential operator of order $(n+1)$ with real constant coefficients. If the characteristic polynomial has only real roots, then $\mathbb{E}$ is an $(n+1)$-dimensional EC-space on the whole of $\mathbb{R}$. If some of the roots are not real, then $\mathbb{E}$ is also an EC-space, however only on sufficiently small intervals, and at least on any interval of length less that π divided by the maximum of the imaginary parts of all non-real roots. Any power functions with pairwise distinct exponents also span EC-spaces, this time on any interval not containing 0.

re:closure **Remark 3.2.** The class of all EC-spaces on a fixed interval I is closed under integration (Rolle's theorem), as well as under multiplication by sufficiently differentiable positive functions. As indicated in the introduction, it is important to observe that it is not closed under the ordinary differentiation D, and this is one of the difficulties encountered in the Chebyshevian world. This is made clear by the classical example of the space spanned by $\mathbb{1}, \cos, \sin$: it is an EC-space on $[a,b]$ if and only if $b-a<2\pi$, while the space spanned by $\cos, \sin$ is an EC-space on $[a,b]$ if and only if $b-a<\pi$.

EC-spaces can be characterised by the existence of special bases resembling polynomial Bernstein bases, see Theorem 3.3 of [16] or Theorem 12 of [17].

th:BLB **Theorem 3.3.** *The space $\mathbb{E}$ is an EC-space on I if and only if for any $c,d \in I$, $c<d$, $\mathbb{E}$ possesses a Bernstein-like basis relative to (c,d), that is, a basis $(B_0, \dots B_n)$ such that for each $i = 0, \dots, n$, B_i vanishes exactly i times at c and exactly $(n-i)$ times at d, and is positive on $]c,d[$.*

To complete Remark 3.2, consider again the EC-space $\mathbb{E}$ spanned by $\mathbb{1}, \cos, \sin$, this time on $[0,\pi]$. The functions

$$B_0(t) := 1+\cos t, \quad B_1(t) := \sin t, \quad B_2(t) := 1-\cos t,$$

form a Bernstein-like basis of $\mathbb{E}$ relative to $(0,\pi)$. All other Bernstein-like bases relative to $(0,\pi)$ are of the form $(\alpha_0 B_0, \alpha_1 B_1, \alpha_2 B_2)$ with any positive $\alpha_0, \alpha_1, \alpha_2$. None of them is normalised. This is the reason why the space $\mathbb{E}$ cannot be used for CAGD purposes on $[0,\pi]$. Actually, as recalled below, this is connected to that fact that the space $D\mathbb{E}$ fails to be an EC-space on $[0,\pi]$ (see Theorem 4.2 of [16]).

th:BB **Theorem 3.4.** *Assume that $\mathbb{E}$ is an $(n+1)$-dimensional EC-space on I. The following properties are equivalent:*

(i) *for any* $c, d \in I$, $c < d$, $\mathbb{E}$ *possesses a Bernstein basis relative to* (c, d), *that is, a Bernstein-like basis relative to* (c, d) *which is normalised;*

(ii) *the space* $\mathbb{E}$ *contains constants and the (n-dimensional) space* $D\mathbb{E}$ *is an EC-space on* I.

Definition 3.5. An EC-space on I is said to be *good for design* when it satisfies (ii) of Theorem 3.4.

Behind this definition is hidden the fact that condition (ii) of Theorem 3.4 characterises the existence of blossoms in the space $\mathbb{E}$. Blossoms are functions of n variables generalising polynomial blossoms. However, unlike them they are not defined by algebraic properties, but in a geometrical way by means of intersections of osculating flats [27, 17]. As soon as they exist, blossoms permit the development of the classical CAGD algorithms like polynomial blossoms, and they guarantee that the Bernstein basis $(B_0, \dots, B_n)$ relative to any given $c, d \in I$, $c < d$, is totally positive on $[c, d]$ (*i.e.*, for any $c \le x_0 < x_1 < \dots < x_n \le d$, all minors of the matrix $(B_i(x_j))_{0 \le i,j \le n}$ are non-negative). It is even the *optimal normalised totally positive basis* of $\mathbb{E}$ restricted to $[c, d]$ (that is, the normalised B-basis in the sense of [3]), see [15, 16, 17]. In order to facilitate the reading we will avoid saying more on blossoms, though they are strongly involved in many proofs.

Example 3.6. Following from Remark 3.2 we can say that the trigonometric space spanned by $\mathbb{1}, \cos, \sin$ is an EC-space good for design only on any interval $[a, b]$ such that $0 < b - a < \pi$.

We now have to recall a classical procedure to build EC-spaces on I, see [30]. This will complete and explain the presentation given in the introduction. Start with a *system* $(w_0, \dots, w_n)$ *of weight functions on* I, *i.e.*, for each i, $w_i \in C^{n-i}(I)$ and is positive on I. The *generalised derivatives* associated with this system are obtained by alternating division by a weight function and ordinary differentiation as follows:

$$L_0 F := \frac{F}{w_0}, \quad L_i F := \frac{D L_{i-1} F}{w_i}, \ i = 1, \dots, n. \tag{3.1}$$

For each $i \le n$, L_i is a linear differential operator of order i. Due to the class of all EC-spaces on I being closed under integration and multiplication by positive functions (see Remark 3.2), the set of all functions $F \in C^n(I)$ for which $L_n F$ is constant on I is an $(n+1)$-dimensional EC-space on I, of which a basis is formed by the generalised monomials (1.4). We denote it as $\mathbb{E} = EC(w_0, \dots, w_n)$. As an instance, if $I = \mathbb{R}$ and $w_i := \mathbb{1}$ for $i = 0, \dots, n$, the generalised derivatives are simply the ordinary derivatives and the space $EC(w_0, \dots, w_n)$ is simply the space of all polynomials of degree at most n. From the recursive definition of generalised derivatives (3.1) one can derive that $w_0 \in EC(w_0, \dots, w_n)$ and that $D\left(EC(\mathbb{1}, w_1, \dots, w_n)\right) = EC(w_1, \dots, w_n)$. Hence, any space EC-space of the form $EC(1\!\!1, w_1, \ldots, w_n)$ is good for design on $[a, b]$. Within the class of all EC-spaces on I which are defined by means of systems of weight functions, the generalised derivatives are appropriate tools to diminish the dimension. However, if the interval is not closed and bounded this class is strictly contained in the class of all EC-spaces on I. By contrast, both classes coincide if the interval is closed and bounded. This is the case surveyed in the subsection below.

3.2. Weight functions and Bernstein operators on closed bounded intervals

In this section we assume that the interval I is a closed bounded interval $[a, b]$. It is known that each EC-space on $[a, b]$ can be associated with systems of weight functions via the procedure described in the previous subsection. We are even able to build all such systems, see [19]. Surprisingly, the search for all such systems is somehow equivalent to the search for all Bernstein operators, as recalled in Theorem 3.10 below.

Unless we explicitly state it differently, we assume that $\mathbb{E} \subset C^n([a, b])$ is an $(n + 1)$-dimensional EC-space good for design on $[a, b]$, in which $(B_0, \ldots, B_n)$ stands for the Bernstein basis relative to (a, b). When no ambiguity is possible, we will omit "relative to (a, b)".

def:BO **Definition 3.7.** Given a strictly increasing sequence $a = \zeta_0 < \zeta_1 < \cdots < \zeta_n = b$, the linear operator $\mathbb{B} : C^0([a, b]) \to \mathbb{E}$ defined by

$$\mathbb{B}F = \sum_{i=0}^{n} F(\zeta_i) B_i, \quad F \in C^0([a, b]), \tag{3.2}$$

is said to be a *Bernstein operator based on* $\mathbb{E}$ if there exists a two-dimensional EC-space $\mathbb{U}$ on $[a, b]$ which is reproduced by $\mathbb{B}$, in the sense that $\mathbb{B}F = F$ for all $F \in \mathbb{U}$.

re:BO **Remark 3.8.** A Bernstein operator $\mathbb{B}$ based on $\mathbb{E}$ reproduces constants. Since it cannot reproduce three linear independent functions (see Proposition 3.7 of [18]), the two-dimensional EC-space reproduced by $\mathbb{B}$ must contain constants. It is thus of the form $\mathbb{E} = \mathrm{span}(1\!\!1, U)$ where U is a strictly increasing function. Determining all Bernstein operators based on $\mathbb{E}$ amounts to determining all non-constant functions they reproduce [18].

Remark 3.9. A Bernstein operator based on $\mathbb{E}$ is a positive operator (*i.e.*, for any non-negative $F \in C^0([a, b])$, the function $\mathbb{B}F$ is non-negative on $[a, b]$). Furthermore, due to the Bernstein basis being totally positive on $[a, b]$, any Bernstein operator based on $\mathbb{E}$ is also variation diminishing [18]. See [10] for further acquaintance with variation diminishing issues.

The following result explains the links between weight functions and Bernstein operators. Its proof, based on Theorem 3.4, strongly involves blossoms and their properties (see Theorem 4.8 of [18] and Theorems 3.2 and 5.1 of [19]).

th1 **Theorem 3.10.** *Given any integer $n \geq 2$, we assume that $\mathbb{E} \subset C^n([a,b])$ is an $(n+1)$-dimensional EC-space good for design on $[a,b]$, and we denote by $(B_0,\ldots,B_n)$ its Bernstein basis (relative to (a,b)). Consider a function $U \in \mathbb{E}$, expanded as $U = \sum_{i=0}^{n} \alpha_i B_i \in \mathbb{E}$, and its first derivative $w_1 := DU$. The following six properties are then equivalent:*

(i) *$\alpha_0,\ldots,\alpha_n$ (i.e., the Bézier points of U relative to (a,b)) form a strictly increasing sequence;*

(ii) *w_1 is positive on $[a,b]$, and there exists a (unique) Bernstein operator based on $\mathbb{E}$ reproducing U;*

(iii) *the function w_1 is positive on $[a,b]$ and, if we define the first order linear differential operator L_1 by $L_1V := (DV)/w_1$, then the n-dimensional space $L_1\mathbb{E}$ is an EC-space good for design on $[a,b]$;*

(iv) *the coordinates of w_1 in any Bernstein-like basis relative to (a,b) of the space $D\mathbb{E}$ are positive;*

(v) *$U(a) < U(b)$ and there exists a nested sequence*

$$\mathbb{E}_1 := \operatorname{span}(\mathbb{1}, U) \subset \mathbb{E}_2 \subset \cdots \subset \mathbb{E}_{n-1} \subset \mathbb{E}_n := \mathbb{E}, \tag{3.3}$$

where, for each $p = 1,\ldots,n$, $\mathbb{E}_p$ is a $(p+1)$-dimensional EC-space on $[a,b]$;

(vi) *the function w_1 is positive on $[a,b]$ and there exists a system $(w_2,\ldots,w_n)$ of weight functions on $[a,b]$ such that $\mathbb{E} = EC(\mathbb{1}, w_1, w_2, \ldots, w_n)$.*

Theorem 3.10 calls for some comments, listed below.

re:zeta **Remark 3.11.** The equivalence between the first two properties in Theorem 3.10 show that, when (i) is satisfied, the Bernstein operator based on $\mathbb{E}$ which reproduces U is given by (3.2) with

$$\zeta_i := U^{-1}(\alpha_i), \quad 0 \leq i \leq n. \tag{3.4}$$

allBO **Remark 3.12.** Each strictly increasing sequence of Bézier points $(\alpha_0,\ldots,\alpha_n)$ generates a unique Bernstein operator based on $\mathbb{E}$. Actually, this establishes a one-to-one correspondence between the set of all Bernstein operators based on $\mathbb{E}$ and the set of all equivalence classes of strictly increasing sequences of $(n+1)$ real numbers under the equivalence relation $(\alpha_0,\ldots,\alpha_n) \sim (\overline{\alpha}_0,\ldots,\overline{\alpha}_n)$ if and only if there exist α, β such that $\overline{\alpha}_i = \alpha\alpha_i + \beta$ for $i = 0,\ldots,n$. Accordingly, as soon as $n \geq 2$, the space $\mathbb{E}$ provides us with infinitely many Bernstein operators. In general none of them has a special meaning for the space $\mathbb{E}$.

:allweightsgood **Remark 3.13.** Iteration of the equivalence (i) $\Leftrightarrow$ (iii) provides us with all systems $(w_1, \dots, w_n)$ of weight functions on $[a, b]$ such that $\mathbb{E} = EC(\mathbb{1}, w_1, \dots, w_n)$. In general none of the associated sequences of generalised derivatives has a special meaning for the space $\mathbb{E}$. Each such system yields a nested sequence (3.3), defined by $\mathbb{E}_i := EC(\mathbb{1}, w_1, \dots, w_i)$, $i = 1, \dots, n$. Conversely, via repeated iteration of (i) $\Rightarrow$ (iii), each sequence (3.3) leads to a system of weight functions $(w_1, w_2, \dots, w_n)$ such that $\mathbb{E} = EC(\mathbb{1}, w_1, w_2, \dots, w_n)$, each w_i being unique up to multiplication by a positive constant. In particular, the class of all $(n+1)$-dimensional EC-spaces which are good for design on $[a, b]$ coincides with the class of all spaces of the form $EC(\mathbb{1}, w_1, \dots, w_n)$, where $(w_1, \dots, w_n)$ ranges over the set of all systems of weight functions on $[a, b]$. By differentiation it also follows that the class of all $(n+1)$-dimensional EC-spaces on $[a, b]$ coincides with the class of all spaces of the form $EC(w_0, w_1, \dots, w_n)$.

re:nestedBO **Remark 3.14.** Assume that $\mathbb{E} \subset \mathbb{E}^*$, where $\mathbb{E}^*$ is an $(n+2)$-dimensional EC-space good for design on $[0, 1]$. Let $\mathbb{E}_1$ be a two-dimensional EC-space which is reproduced by a Bernstein operator based on $\mathbb{E}$. From the existence of a nested sequence (3.3) we can assert that $\mathbb{E}_1$ is automatically reproduced too by a Bernstein operator based on $\mathbb{E}^*$.

We conclude this section with the subsequent observation: the three properties (iii) and (iv), and (vi) of Theorem 3.10 can be considered as properties of the EC-space $D\mathbb{E}$. We can thus restate them as follows:

th:w0 **Theorem 3.15.** *Let $\mathbb{E} \subset C^n([a, b])$ be an $(n+1)$-dimensional EC-space on $[a, b]$ (not necessarily good for design). Given $w_0 \in \mathbb{E}$, the following properties are equivalent:*

(i) *the coordinates of w_0 in any given Bernstein-like basis of $\mathbb{E}$ relative to (a, b) are all positive;*

(ii) *w_0 is positive on $[a, b]$ and the space $L_0\mathbb{E}$ obtained by division by w_0 is good for design;*

(iii) *there exists a system $(w_1, \dots, w_n)$ of weight functions on $[a, b]$ such that $\mathbb{E} = EC(w_0, w_1, \dots, w_n)$.*

re:BBab **Remark 3.16.** Theorem 3.4 says that, among all EC-spaces containing constants on a given interval I, those which are good for design can be characterised by the presence of Bernstein bases relative to any pair of distinct points of I. On a closed bounded interval $[a, b]$, one can more simply characterise them by the presence of a Bernstein basis relative to (a, b). This is obtained when applying (i) $\Rightarrow$ (ii) of Theorem 3.15 with $w_0 := \mathbb{1}$.

re:BLO **Remark 3.17.** When the $(n+1)$-dimensional EC-space $\mathbb{E}$ is not assumed to be good for design on $[a, b]$, instead of Bernstein operators one can introduce *Bernstein-like operators* based on $\mathbb{E}$. Such an operator is associated with a unique two-dimensional EC-space $EC(w_0, w_1)$ which it reproduces.

However there are specific difficulties due to the fact that Bernstein-like bases are unique only up to multiplication by positive numbers. Therefore, given any w_0 satisfying (i) of Theorem 3.15, it is more convenient to define the Bernstein-like operator $\overline{\mathbb{B}}$ reproducing $EC(w_0, w_1)$ as

$$L_0\left(\overline{\mathbb{B}}F\right) := \mathbb{B}\left(L_0 F\right), \quad F \in C^0([a,b]),$$

where $\mathbb{B}$ is the unique Bernstein operator based on $L_0\mathbb{E}$ which reproduces $EC(\mathbb{1}, w_1)$. We will not go further into this matter, preferring to refer the reader to [18].

§4. View $\mathbb{P}_n$ as an EC-space good for design on $[0,1]$

In the present section as well as in the following one, we apply recently obtained results recalled in Section 3 to polynomial spaces on a closed bounded interval, say on $[0,1]$ without loss of generality. We are thus revisiting the space $\mathbb{P}_n$, viewing it as an instance of EC-space on $[0,1]$. This revisit will prove to be fruitful since all results we will deduce from it concerning Bernstein operators, either polynomial (present section) or rational (next section) are new.

In the present section, we actually consider $\mathbb{P}_n$ as an element of the class of all EC-spaces good for design on $[0,1]$, which corresponds to writing $\mathbb{P}_n$ under the general form

$$\mathbb{P}_n = EC(\mathbb{1}, w_1, \ldots, w_n), \tag{4.1}$$

where $(w_1, \ldots, w_n)$ is an appropriate system of weight functions on $[0,1]$. Any such equality should be understood as the replacement of the ordinary derivatives (corresponding to the equality $\mathbb{P}_n = EC(\underbrace{\mathbb{1}, \mathbb{1}, \ldots, \mathbb{1}}_{(n+1)\text{ times}})$) by generalised ones.

4.1. Polynomial Bernstein operators

As an EC-space good for design on $[0,1]$, we can apply Theorem 3.10 to the polynomial space $\mathbb{P}_n$. The classical Bernstein operator (2.2) is a Bernstein operator in the sense of Definition 3.7 since it reproduces $\mathbb{P}_1$. This corresponds to the fact that the identity X_1 has strictly increasing Bézier points $\frac{i}{n}$, $i = 0, \ldots, n$. On account of all results and remarks in the previous section, we can state:

Theorem 4.1. *For each $n \geq 2$, there exist infinitely many polynomial Bernstein operators of degree n (that is, infinitely many Bernstein operators based on the space $\mathbb{P}_n$), characterised by the two-dimensional EC-space they reproduce, among which the classical Bernstein operator (2.2) is the one which reproduces $\mathbb{P}_1$.*

Remark 4.2. We thus observe that building all polynomial Bernstein operators of degree n can exactly be viewed as searching for all weight functions w_1 permitting an equality of the form (4.1). In particular $w_1 = \mathbb{I}$ corresponds to the classical Bernstein operator (2.2). It also amounts to finding all first order differential operators L_1, obtained by composition of the ordinary derivative and division by a positive function, which, applied to $\mathbb{P}_n$, ensures the same property as the ordinary derivative D, *i.e.*, decrease the dimension by one within the class of all EC-spaces good for design on $[0, 1]$.

The use of generalised derivatives or of polynomial Bernstein operators different from the classical ones may seem surprising. It is therefore certainly useful to give a few detailed examples.

Example 4.3. Assume that $n = 3$ and consider the polynomial function

$$U(t) = \sum_{i=0}^{3} u_i B_i^3(t) = B_0^3(t) + 2B_1^3(t) + 4B_2^3(t) + 7B_3^3(t) = 3t^2 + 3t + 1, \quad t \in [0, 1].$$

The sequence $(u_0, u_1, u_2, u_3) = (1, 2, 4, 7)$ of its Bézier points being strictly increasing, we know that there exists a unique polynomial Bernstein operator $\mathbb{B}$ which reproduces U. In order to compute $\mathbb{B}F = \sum_{i=0}^{3} F(\zeta_i)B_i^3$ we have to solve the equations $U(\zeta_i) = u_i$ for $i = 0, \ldots, 3$. This yields

$$\mathbb{B}F := F(0)B_0^3 + F\left(\frac{-3+\sqrt{21}}{6}\right) B_1^3 + F\left(\frac{-1+\sqrt{5}}{2}\right) B_2^3 + F(1)B_3^3, \quad F \in C^0([0, 1]).$$

Note that, via Theorem 3.10, this operator $\mathbb{B}$ corresponds to all possible equalities of the form

$$\mathbb{P}_3 = EC(\mathbb{I}, w_1, w_2, w_3), \quad \text{with } w_1(t) := 2t + 1 \text{ for all } t \in [0, 1].$$

Example 4.4. Let us consider the polynomial function $U \in \mathbb{P}_n$ defined by

$$U(t) := 2t^3 - 3t^2 + 3t - 1 = \sum_{i=0}^{3} \alpha_i B_i^3(t) = -B_0^3(t) + B_3^3(t), \quad t \in [0, 1].$$

Since $U'(t) = 3\left(t^2 + (t-1)^2\right)$, the function $w_1 := U'$ is positive on $[0, 1]$. We can thus introduce the first order differential operator $L_1V := \frac{DV}{w_1}$. The three-dimensional space $L_1\mathbb{P}_3$, obtained by division of $\mathbb{P}_2$ by a positive C^∞ function, is an EC-space on $[0, 1]$ (see Remark 3.2). However, it is not good for design on $[0, 1]$, *i.e.*, the two-dimensional space $DL_1\mathbb{P}_3$ is not an EC-space on $[0, 1]$. In other words, L_1 is not a generalised derivative in $\mathbb{P}_3$, which means that we cannot find any system (w_2, w_3) of weight functions on $[0,1]$ permitting to write $\mathbb{P}_3 = EC(\mathbb{1}, w_1, w_2, w_3)$. This is due to the sequence $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (-1, 0, 0, 1)$ not being strictly increasing. On the other hand, U being strictly increasing, clearly we cannot expect to find $0 < \zeta_1 < \zeta_2 < 1$ such that $U(\zeta_i) = \alpha_i = 0$ for $i = 1, 2$. Hence, the function U cannot be reproduced by a Bernstein operator based on $\mathbb{P}_3$. Nevertheless, out of dimension elevation we can write U as

$$U = -B_0^4 - \frac{1}{4}B_1^4 + \frac{1}{4}B_3^4 + B_4^4.$$

The Bézier points of U viewed as an element of $\mathbb{P}_4$ thus form a strictly increasing sequence. Therefore, U is reproduced by a Bernstein operator based on $\mathbb{P}_4$. Equivalently, we can say that there exists systems (w_2, w_3, w_4) of weight functions on $[0,1]$ such that $\mathbb{P}_4 = EC(\mathbb{1}, w_1, w_2, w_3, w_4)$. In other words, the four-dimensional EC-space $L_1\mathbb{P}_4$ is good for design on $[0,1]$, that is, $DL_1\mathbb{P}_4$ is a three-dimensional EC-space on $[0,1]$. Accordingly, L_1 is indeed a generalised derivative in $\mathbb{P}_4$. From Theorem 3.10 we can also say that it is possible to find infinitely many EC-spaces $\mathbb{E}_2$, $\mathbb{E}_3$, of dimension 3, 4, respectively, such that

$$\mathrm{span}(\mathbb{1}, U) \subset \mathbb{E}_2 \subset \mathbb{E}_3 \subset \mathbb{P}_4,$$

but none of them is included in $\mathbb{P}_3$.

4.2. Convergence of sequences of polynomial Bernstein operators

According to Remark 3.14, a two-dimensional EC-space $\mathbb{E}_1$ which is reproduced by a polynomial Bernstein operator of degree n will automatically be reproduced too by a polynomial Bernstein operator of any degree $k \geq n$. This observation gives sense to considering sequences of polynomial Bernstein operators all reproducing the same two-dimensional EC-space. This is what we do in Theorem 4.5 below which proves that convergence in the sense of Theorem 2.2 is not specific to the classical Bernstein operators (2.2).

th:CONV **Theorem 4.5.** *Any infinite sequence of polynomial Bernstein operators of increasing degrees which all reproduce the same two-dimensional EC-space on* $[0,1]$ *permits uniform approximation of any continuous function on* $[0,1]$.

Proof. We start with a function $U \in \mathbb{P}_n$ for some $n \geq 1$, assumed to have strictly increasing Bézier points $u_{n,0}, \ldots, u_{n,n}$. We know the existence of a nested sequence

$$\mathbb{E}_1 := \mathrm{span}(\mathbb{1}, U) \subset \mathbb{E}_2 \subset \cdots \subset \mathbb{E}_n := \mathbb{P}_n \subset \mathbb{P}_{n+1} \subset \cdots,$$

As observed in Remark 3.14, for any $k \geq n$, U is reproduced by a polynomial Bernstein operator $\mathbb{B}_k$ of degree k. We want to prove that the sequence $\mathbb{B}_k$, $k \leq n$, satisfies

$$\lim_{k\to+\infty} \|F - \mathbb{B}_k F\|_\infty = 0 \quad \text{for any } F \in C^0([0,1]). \tag{4.2}$$

The case $n = 1$ gives the sequence of classical polynomial Bernstein operators $\mathbb{B}_k^*$, $k \geq 1$. We can therefore assume that $n \geq 2$. Since each $\mathbb{B}_k$, $k \geq n$, reproduces the two functions $\mathbb{1}, U$, Korovkin's theorem ensures that it is sufficient to prove (4.2) for any $V \in \mathbb{E}_2$. Actually, we will more generally select any $V \in \mathbb{P}_n$ and prove (4.2) for this V.

Let us expand U and V in the Bernstein bases as

$$U = \sum_{i=0}^{k} u_{k,i} B_i^k, \quad V = \sum_{i=0}^{k} v_{k,i} B_i^k, \quad k \geq n.$$

For each $k \geq n$, the polynomial Bernstein operator $\mathbb{B}_k$ is defined on $C^0([0,1])$ by

$$\mathbb{B}_k F = \sum_{i=0}^{k} F(t_{k,i}) B_i^k, \quad \text{with } U(t_{k,i}) := u_{k,i} \text{ for } i = 0, \dots, k.$$

We thus obtain, for any $k \geq n$,

$$\|\mathbb{B}_k V - V\|_\infty = \left\| \sum_{i=0}^{k} \left[V(t_{k,i}) - v_{k,i}\right] B_i^k \right\|_\infty \leq \max_{0\leq i\leq k} |V(t_{k,i}) - v_{k,i}| .$$

In order to make sure that $\lim_{k\to+\infty} \|V - \mathbb{B}_k V\|_\infty = 0$, it is sufficient to prove that

$$\lim_{k\to+\infty} \max_{0\leq i\leq k} |V(t_{k,i}) - v_{k,i}| = 0. \tag{4.3}$$

Now, from Proposition 2.3 we know that

$$\lim_{k\to+\infty} \max_{0\leq i\leq k} \left|U(t_{k,i}^*) - u_{k,i}\right| = 0 = \lim_{k\to+\infty} \max_{0\leq i\leq k} \left|V(t_{k,i}^*) - v_{k,i}\right| . \tag{4.4}$$

The leftmost equality can be replaced by

$$\lim_{k\to+\infty} \max_{0\leq i\leq k} \left|U(t_{k,i}^*) - U(t_{k,i})\right| = 0. \tag{4.5}$$

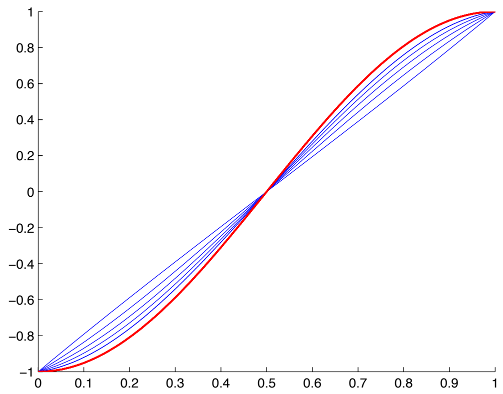

Figure 1: The function F defined in (4.6) (red graph), and its approximant $\mathbb{B}_3 F$, where $\mathbb{B}_3$ is the polynomial Bernstein operator of degree 3 reproducing U_α defined in (4.7), with $\alpha := \frac{i}{6}$. From top to bottom in the left part of the graphs, i =1; 2 (*i.e.*, $\mathbb{B}_3 = \mathbb{B}_3^*$); 3; 4; 5.

The function U being strictly increasing on $[0,1]$, we can consider its inverse U^{-1}. Using the uniform continuity of $V \circ U^{-1}$ on the interval $[U(0), U(1)]$, (4.5) implies that

$$\lim_{k\to+\infty} \max_{0\le i\le k} \left| V \circ U^{-1}\big(U(t^*_{k,i})\big) - V \circ U^{-1}\big(U(t_{k,i})\big) \right| = \lim_{k\to+\infty} \max_{0\le i\le k} \left| V(t^*_{k,i}) - V(t_{k,i}) \right| = 0.$$

To obtain (4.3) we just have to additionally use the rightmost part of (4.4) after observing that

$$|V(t_{k,i}) - v_{k,i}| \le \left| V(t_{k,i}) - V(t^*_{k,i}) \right| + \left| V(t^*_{k,i}) - v_{k,i} \right|.$$

■

Remark 4.6. The latter result suggests that it might be interesting to do polynomial approximation of continuous functions with shape parameters. The shape parameters would be provided by the strictly increasing sequence $(u_{n,0}, \ldots, u_{n,n})$ we start with. Our purpose is not to thoroughly investigate this question, but rather to briefly illustrate this with an example. Consider the function

$$F(t) := \sin\left(\pi t - \frac{\pi}{2}\right), \quad t \in [0,1]. \tag{4.6}$$

For any $n \geq 3$, let $\mathbb{B}_n$ be the Bernstein operator based on $\mathbb{P}_n$ which reproduces the function U_α depending on the parameter $\alpha \in]0,1[$

$$U_\alpha(t) := -B_0^3(t) - \alpha B_1^3(t) + \alpha B_2^3(t) + B_3^3(t) = (2t-1)\big[1 + (3\alpha - 1)t(1-t)\big]. \tag{4.7}$$

Observe that, for $\alpha = \frac{1}{3}$, $U_\alpha(t) = 2t - 1$. Accordingly, for this specific value of the parameter α, the operator $\mathbb{B}_n$ is the classical Bernstein operator of degree n, that is $\mathbb{B}_n^*$. Figure 1 presents the graphs of the function $\mathbb{B}_3 F$ for various values of the parameter α, by comparison with the graph of the function F itself. Visually speaking, $\mathbb{B}_3 F$ is all the closer to F as α is closer to 1. In particular, the function $\mathbb{B}_3 F$ obtained for $\alpha = \frac{5}{6}$ seems a much better approximant than $\mathbb{B}_3^* F$.

§5. Simply view $\mathbb{P}_n$ as an EC-space on $[0,1]$

In the present section, we forget about the fact that the EC-space $\mathbb{P}_n$ is good for design, that is, about the fact that $D\mathbb{P}_n = \mathbb{P}_{n-1}$ itself is an EC-space on $[0,1]$. In terms of weight functions, this means that we no longer require that the first weight function be equal to $\mathbb{1}$. We are now rather interested in all polynomials $\Omega \in \mathbb{P}_n$ ensuring equalities of the form

$$\mathbb{P}_n = EC(\Omega, w_1, \dots, w_n). \tag{5.1}$$

5.1. Rational spaces and weight functions

In order to fix the notations which will be used subsequently it is worthwhile rewriting Theorem 3.15 as follows:

th:Omega **Theorem 5.1.** *Given $\Omega := \sum_{i=0}^n \omega_i B_i^n \in \mathbb{P}_n$, the following three properties are equivalent:*

(i) *$\omega_i > 0$ for $i = 0, \dots, n$;*

(ii) *Ω is positive on $[a,b]$ and the space $\widehat{\mathbb{P}}_n$ obtained from $\mathbb{P}_n$ by division of all its elements by Ω is an EC-space good for design on $[0,1]$;*

(iii) *there exist systems $(w_1, \dots, w_n)$ of weight functions on $[0,1]$ such that (5.1) holds.*

From now on, the function Ω is fixed, and we assume that all ω's are positive. Then, division of both sides of the equality $\Omega := \sum_{i=0}^{n} \omega_i B_i^n$ by Ω shows that the sequence

$$\widehat{B}_i := \frac{\omega_i B_i^n}{\Omega}, \quad i = 0, \ldots, n,$$

is normalised and presents the same property of zeroes as the classical Bernstein basis $(B_0^n, \ldots, B_n^n)$. Accordingly, it is the Bernstein basis relative to $(0,1)$ in $\widehat{\mathbb{P}}_n$. The space $\widehat{\mathbb{P}}_n$ is thus composed of all functions

$$\widehat{F}(t) = \sum_{i=0}^{n} \alpha_i \frac{\omega_i B_i^n(t)}{\Omega(t)} = \frac{\sum_{i=0}^{n} \alpha_i \omega_i B_i^n(t)}{\sum_{i=0}^{n} \omega_i B_i^n(t)}, \quad t \in [0,1]. \tag{5.2}$$

In other words, (5.2) proves that $\widehat{\mathbb{P}}_n$ *is the rational space of degree* n *based on the positive weights* $\omega_0, \ldots, \omega_n$. To avoid confusion with weight functions, we will rather say that it is *based on* Ω. For a classical approach of rational spaces, see for instance [7, 9, 25].

On account of (i) $\Rightarrow$ (ii), Theorem 5.1 ensures that the rational space $\widehat{\mathbb{P}}_n$ is an EC-space good for design on $[0,1]$. The interesting part of this statement is that

Theorem 5.2. ([22]) *The first derivative* $D\widehat{\mathbb{P}}_n$ *of a rational space* $\widehat{\mathbb{P}}_n$ *is an EC-space on* $[0,1]$.

re:rational **Remark 5.3.** We thus observe that building all rational spaces of degree n can exactly be viewed as searching for all weight functions Ω permitting an equality of the form (5.1). It also amounts to finding all first order differential operators DL_0 ensuring the same property as the ordinary derivative D when applied to $\mathbb{P}_n$, *i.e.*, decrease the dimension by one within the class of all EC-spaces on $[0,1]$.

5.2. Convergence of rational Bernstein operators

In the previous subsection we have seen that the space $\widehat{\mathbb{P}}_n$ is an EC-space good for design on $[0,1]$. As so, it provides us with Bernstein operators based on it, and we now focus on these operators. From Theorem 3.10 we know that they correspond to all weight functions w_1 permitting to write the rational space $\widehat{\mathbb{P}}_n$ as $\widehat{\mathbb{P}}_n = EC(\mathbb{1}, w_1, w_2, \ldots, w_n)$, that is, the polynomial space $\mathbb{P}_n$ itself as $\mathbb{P}_n = EC(\Omega, w_1, w_2, \ldots, w_n)$. From Remark 3.17 we also know that they correspond to all Bernstein-like operators based on $\mathbb{P}_n$ which reproduce a two-dimensional space of the form $EC(\Omega, w_1)$.

We will more precisely be interested in convergence. This is why we will now write Ω as

$$\Omega = \sum_{i=0}^{k} \omega_{k,i} B_i^k \quad \text{for all } k \geq n,$$

so that in particular $\omega_{n,i} = \omega_i$ for $i = 0, \dots, n$. Applying the degree elevation process to Ω shows that, for each $k \geq n$, and each $i = 1, \dots, k$, $\omega_{k+1,i}$ is a strictly convex combination of $\omega_{k,i-1}$, $\omega_{k,i}$. Moreover $\omega_{k+1,0} = \omega_{k,0}$ and $\omega_{k+1,k+1} = \omega_{k,k}$. By induction we can therefore state that, for each $k \geq n$, the Bézier points of Ω viewed as an element of $\mathbb{P}_k$ can be expressed as convex combinations of the initial positive Bézier points $\omega_0, \dots, \omega_n$. Hence they all are positive. Accordingly, the nested sequence

$$\widehat{\mathbb{P}}_n = \frac{1}{\Omega}\mathbb{P}_n \subset \widehat{\mathbb{P}}_{n+1} := \frac{1}{\Omega}\mathbb{P}_{n+1} \subset \cdots \subset \widehat{\mathbb{P}}_k := \frac{1}{\Omega}\mathbb{P}_k \subset \widehat{\mathbb{P}}_{k+1} := \frac{1}{\Omega}\mathbb{P}_{k+1} \subset \cdots \tag{5.3}$$

is a nested sequence of rational spaces, all based on the same function Ω. According to Remark 3.14, a two-dimensional EC-space which is reproduced by a rational Bernstein based on $\widehat{\mathbb{P}}_n$ will automatically be reproduced too by a rational Bernstein operator based on $\widehat{\mathbb{P}}_k$ for all $k \geq n$. This situation (5.3) is the most natural one to build infinite sequences of rational Bernstein operators, as will be done in the theorem below.

CONVRat **Theorem 5.4.** *Consider an infinite sequence $\widehat{\mathbb{B}}_k$, $k \geq n$, of Bernstein operators respectively based on the rational spaces $\widehat{\mathbb{P}}_k$, $k \geq n$, introduced in (5.3), assumed to all reproduce the same two-dimensional EC-space on $[0,1]$. Then, for each $F \in C^0([0,1])$, the sequence $\widehat{\mathbb{B}}_k F$, $k \geq n$, converges to F uniformly on $[0,1]$.*

Proof. Let $\widehat{\mathbb{E}}_1$ be the two-dimensional EC-space on $[0,1]$ which is reproduced by each operator $\widehat{\mathbb{B}}_k$, $k \geq n$. For any $k \geq n$, let $(\widehat{B}_{k,0}, \dots, \widehat{B}_{k,k})$ denote the Bernstein basis of the rational space $\widehat{\mathbb{P}}_k$. In $\widehat{\mathbb{E}}_1$ we select a strictly increasing function $\widehat{U}$, expanded as

$$\widehat{U} = \sum_{i=0}^{k} \widehat{u}_{k,i} \widehat{B}_{k,i}, \quad k \geq n.$$

For each $k \geq n$, the Bézier points $\widehat{u}_{k,0}, \dots, \widehat{u}_{k,k}$ of $\widehat{U}$ form a strictly increasing sequence, and the Bernstein operator $\widehat{\mathbb{B}}_k$ based on the rational space $\widehat{\mathbb{P}}_k$ which reproduces $\mathbb{E}_1$ is given by

$$\widehat{\mathbb{B}}_k F := \sum_{i=0}^{k} F(\widehat{t}_{k,i}) \widehat{B}_{k,i}, \quad \text{with } \widehat{U}(\widehat{t}_{k,i}) = \widehat{u}_{k,i} \text{ for } i = 0, \dots, k.$$

All positive operators $\widehat{\mathbb{B}}_k$, $k \geq n$, reproduce $\mathbb{1}$ and $\widehat{U}$. According to Korovkin's theorem, in order to prove the claimed result, it suffices to show that $\lim_{k \to +\infty} \left\| \widehat{\mathbb{B}}_k V - V \right\|_\infty = 0$ for one given $V \in$

$C^0([0,1])$ chosen so that the functions $\mathbb{1}, \widehat{U}$, and V span a three-dimensional EC-space on $[0,1]$ (see, for instance, [5], Theorem 4.2). Without loss of generality, we can assume that $n \geq 2$ and prove the desired property for any $\widehat{P} \in \widehat{\mathbb{P}}_n$. Select any such $\widehat{P}$ in the rational space $\widehat{\mathbb{P}}_n$, expanded as

$$\widehat{P} = \frac{P}{\Omega} = \sum_{i=0}^{k} \widehat{p}_{k,i} \widehat{B}_{k,i} = \sum_{i=0}^{k} \frac{p_{k,i}}{\omega_{k,i}} \widehat{B}_{k,i}, \quad \text{with } P = \sum_{i=0}^{k} p_{k,i} B_{k,i}, \quad k \geq n. \tag{5.4}$$

Clearly, it is sufficient to prove that

$$\lim_{k \to +\infty} \max_{0 \leq i \leq k} \left| \widehat{P}(\widehat{t}_{k,i}) - \widehat{p}_{k,i} \right| = 0. \tag{5.5}$$

Now, applying Proposition 2.3 both to P and Ω, we can assert that

$$\lim_{k \to +\infty} \max_{0 \leq i \leq k} \left| P(t^*_{k,i}) - p_{k,i} \right| = 0 = \lim_{k \to +\infty} \max_{0 \leq i \leq k} \left| \Omega(t^*_{k,i}) - \omega_{k,i} \right|. \tag{5.6}$$

Using (5.4) we can write

$$\widehat{P}(t^*_{k,i}) - \widehat{p}_{k,i} = \frac{P(t^*_{k,i})}{\Omega(t^*_{k,i})} - \frac{p_{k,i}}{\omega_{k,i}} = \frac{\omega_{k,i} \left[P(t^*_{k,i}) - p_{k,i} \right] + p_{k,i} \left[\omega_{k,i} - \Omega(t^*_{k,i}) \right]}{\omega_{k,i} \Omega(t^*_{k,i})}. \tag{5.7}$$

As already observed, for each $k \geq n$, and for each $i = 0, \ldots, k$, $\omega_{k,i}$ can be calculated as a convex combination of $\omega_{n,0}, \ldots, \omega_{n,n}$ and each $p_{k,i}$ can be calculated via the same convex combination of $p_{n,0}, \ldots, p_{n,n}$. It follows that, for each $k \geq n$ and each $i = 0, \ldots, k$, we have

$$\begin{gathered} 0 < m := \min(\omega_{n,0}, \ldots, \omega_{n,n}) \leq \omega_{k,i} \leq M := \max(\omega_{n,0}, \ldots, \omega_{n,n}), \\ |p_{k,i}| \leq Q := \max(|p_{n,0}|, \ldots, |p_{n,n}|). \end{gathered} \tag{5.8}$$

On the other hand, for any $t_1, \ldots, t_n \in [0,1]$, the value $\omega_n(t_1, \ldots, t_n)$ of the blossom ω_n of $\Omega \in \mathbb{P}_n$ can be calculated as a convex combination of the Bézier points $\omega_{n,0}, \ldots, \omega_{n,n}$ of Ω, via an n-step generalised version of the de Casteljau evaluation algorithm. This holds in particular for each $\Omega(t) = \omega_n(t^{[n]})$, $t \in [0,1]$. In other words we can also state that

$$0 < m \leq \Omega(t) \leq M \quad \text{for all } t \in [0,1]. \tag{5.9}$$

Taking (5.8) and (5.9) into account, (5.7) leads to

$$\left|\widehat{P}(t^*_{k,i}) - \widehat{p}_{k,i}\right| \leq \frac{M \max\limits_{0\leq j\leq k}\left|P(t^*_{k,j}) - p_{k,j}\right| + Q \max\limits_{0\leq i\leq k}\left|\Omega(t^*_{k,i}) - \omega_{k,i}\right|}{m^2},$$

thus proving that

$$\lim_{k\to+\infty}\max_{0\leq i\leq k}\left|\widehat{P}(t^*_{k,i}) - \widehat{p}_{k,i}\right| = 0. \tag{5.10}$$

In particular, for $\widehat{P} = \widehat{U}$, this yields

$$\lim_{k\to+\infty}\max_{0\leq i\leq k}\left|\widehat{U}(t^*_{k,i}) - \widehat{U}(\widehat{t}_{k,i})\right| = 0. \tag{5.11}$$

As in the proof of Theorem 4.5, uniform continuity arguments eventually lead to the fact that

$$\lim_{k\to+\infty}\max_{0\leq i\leq k}\left|\widehat{P}(\widehat{t}_{k,i}) - \widehat{P}(t^*_{k,i})\right| = 0.$$

This proves (5.5) via an appropriate triangular inequality and (5.10). ■

nsion elevation

Remark 5.5. Generalising Remark 2.4, observe that the convergence of the so-called *dimension elevation algorithm* generated by the infinite sequence (5.3) can efficiently be obtained as a consequence of (5.5). Subsequently, we use exactly the same notations as in Theorem 5.4 and in its proof. Given any $\widehat{P} \in \widehat{\mathbb{P}}_j$ for any given $j \geq n$, consider the sequence $\widehat{L}_k\widehat{P}$, $k \geq j$, defined by

$$\widehat{L}_k\widehat{P}(\widehat{t}_{k,i}) = \widehat{p}_{k,i} \text{ for } i = 0,\dots,k, \quad \widehat{L}_k\widehat{P} \text{ is affine on } \left[\widehat{t}_{k,i}, \widehat{t}_{k,i+1}\right] \text{ for } i = 0,\dots,k-1,$$

where, for each $k \geq j$, $\widehat{p}_{k,i}$, $i = 0,\dots,k$, are the Bézier points of $\widehat{P}$ in $\widehat{\mathbb{P}}_k$. The set of all $\widehat{t}_{k,i}$, $k \geq n$, $0 \leq i \leq k$, being dense in $[0,1]$ (see (5.11)), convergence of the dimension elevation algorithm can be viewed as the uniform convergence of the sequence $\widehat{L}_k\widehat{P}$, $k \geq n$, to $\widehat{P}$ on $[0,1]$, for any such $\widehat{P}$. Select a nested sequence $\widehat{\mathbb{E}}_2,\dots,\widehat{\mathbb{E}}_{n-1}$ of EC-spaces on $[0,1]$ of increasing dimension, so that $\widehat{\mathbb{E}}_1 \subset \widehat{\mathbb{E}}_2 \subset \cdots \subset \widehat{\mathbb{E}}_{n-1} \subset \widehat{\mathbb{P}}_n$. This is made possible by Theorem 3.10. Setting $\widehat{\mathbb{E}}_k := \widehat{\mathbb{P}}_k$ for $k \geq n$, for any $j \geq 2$, we can then choose a function $\widehat{U}_j \in \widehat{\mathbb{E}}_j \setminus \widehat{\mathbb{E}}_{j-1}$ so that $\widehat{U}_j$ will be $\widehat{U}$-convex, that is, so that the function $\widehat{U}_j \circ \widehat{U}^{-1}$ will be convex on $[\widehat{U}(0), \widehat{U}(1)]$. Out of linearity, it suffices to prove uniform convergence for each $\widehat{U}_j$, $j \geq 2$. Taking the $\widehat{U}$-convexity into account we have (see [18])

$$\widehat{\mathbb{B}}_k\widehat{U}_j \geq \widehat{L}_k\left(\widehat{\mathbb{B}}_k\widehat{U}_j\right) \geq \widehat{U}_j \geq \widehat{L}_k\widehat{U}_j, \quad k \geq j.$$

Similar arguments as those in Remark 2.4 enable us to state that $\lim_{k\to+\infty}\|\widehat{U}_j - \widehat{L}_k\widehat{U}_j\|_\infty = 0$.

5.3. An example

Any $(n+1)$-dimensional EC-space $\mathbb{E}$ assumed to be good for design on $[0,1]$ permits to obtain any straight line segments in $\mathbb{R}^d$, $d \geq 2$, as examples of parametric curves: it suffices to choose any Bézier points in a monotonic way on the segment in question. However such segments cannot be described as graphs of a function with values in $\mathbb{R}^{d-1}$, unless $\mathbb{P}_1 \subset \mathbb{E}$. This property is generally referred to as *linear precision*. The question of linear precision for rational spaces was solved in [6]; also see [8]. Linear precision can be useful in many ways, and this is why we devote this section to it. For instance, recently rational spaces with linear precision were successfully used to build rational shape-preserving Hermite interpolants in [23].

As a preliminary result we need to recall the corresponding obvious observation in order to make forthcoming formulæ natural. Since $\widehat{\mathbb{P}}_n$ contains constants, it satisfies linear precision if and only if the identity belongs to $\widehat{\mathbb{P}}_n$. The very definition of the space $\widehat{\mathbb{P}}_n$ makes the following statement obvious.

Lemma 5.6. *The rational space $\widehat{\mathbb{P}}_n$ satisfies linear precision (i.e., $\mathbb{P}_1 \subset \widehat{\mathbb{P}}_n$) if and only if Ω is of degree at most $(n-1)$.*

Let us come back to the infinite nested sequence of rational spaces introduced in (5.3). Clearly, if $\widehat{\mathbb{P}}_n$ satisfies linear precision, so does any $\widehat{\mathbb{P}}_k$, $k \geq n$. Nevertheless, it should be observed that, in case $\widehat{\mathbb{P}}_n$ fails to satisfy linear precision, *i.e.*, in case Ω is of exact degree n, then each further space $\widehat{\mathbb{P}}_k$, $k \geq n+1$, does satisfy linear precision all the same. This is why, for the rest of this section, without loss of generality we do assume that Ω is of degree at most $(n-1)$. Equivalently, we assume the existence of real numbers $\omega_0^*, \dots, \omega_{n-1}^*$ (the Bézier points of Ω as an element of $\mathbb{P}_{n-1}$) leading to the positive $\omega_0, \dots, \omega_n$ via degree elevation from $\mathbb{P}_{n-1}$ to $\mathbb{P}_n$, *i.e.*, ensuring that

$$\omega_0 := \omega_0^*, \quad \omega_i := \frac{i}{n}\omega_{i-1}^* + \frac{n-i}{n}\omega_i^* \quad \text{for } 1 \leq i \leq n-1, \quad \omega_n := \omega_{n-1}^*. \tag{5.12}$$

Remark 5.7. Note that, in order to ensure the positivity of all $\omega_0, \dots, \omega_n$, it is sufficient to require that $\omega_0^*, \dots, \omega_{n-1}^*$ be positive. However, this is not necessary. For instance, for $n = 3$, the choice $\omega_0^* = \omega_2^* = 2$, $\omega_1^* = -\frac{1}{2}$, yields $\omega_2 = \omega_3 = \frac{1}{3}$. Accordingly, though the space $\widehat{\mathbb{P}}_{n-1} := \frac{1}{\Omega}\mathbb{P}_{n-1}$ is well defined due to Ω being positive on $[0,1]$, this space is not necessarily a rational space of degree $(n-1)$.

The problem we want to tackle here is the following one: how should we select $\omega_0^*, \dots, \omega_{n-1}^*$ so that the nested sequence (5.3) provides us with a corresponding sequence $\widehat{\mathbb{B}}_k$, $k \geq n$, of Bernstein operators all reproducing the space $\mathbb{P}_1$? From Remark 3.14 we know that it is sufficient to ensure reproduction of $\mathbb{P}_1$ by a Bernstein operator based on $\widehat{\mathbb{P}}_n$. From Theorem 3.10 we know that this holds if and only if the Bézier points of the identity X_1 as an element of $\widehat{\mathbb{P}}_n$ form a strictly increasing sequence. To solve this question, we start with the following simple lemma on polynomial blossoms.

le:blosprod **Lemma 5.8.** *Given any integers $n_1, n_2 \geq 1$, and for $i = 1, 2$, given $P_i \in \mathbb{P}_{n_i}$, let $P \in \mathbb{P}_n$ be defined by $P := P_1 P_2$, with $n := n_1 + n_2$. Then, the blossom p in n variables of P is given by*

$$p(t_1, \ldots, t_n) = \frac{1}{\binom{n}{n_1}} \sum p_1(t_{i_1}, \ldots, t_{i_{n_1}}) p_2(t_{j_1}, \ldots, t_{j_{n_2}}), \quad t_1, \ldots, t_n \in [0, 1], \tag{5.13}$$

where, for $i = 1, 2$, p_i denotes the blossom in n_i variables of $P_i \in \mathbb{P}_{n_i}$, and where the sum in (5.13) is taken over all sequences of indices $1 \leq i_1 < \cdots < i_{n_1} \leq n$, $1 \leq j_1 < \cdots < j_{n_2} \leq n$ with $i_r \neq j_s$ for $1 \leq r \leq n_1$, $1 \leq s \leq n_2$.

Proof. Due to the properties of the blossoms p_i, $i = 1, 2$, the function appearing in the right-hand side of (5.13) is clearly symmetric and affine in each variable $t_1, \ldots, t_n$, and clearly too, on the diagonal of $[0, 1]^n$ we have $p(t^{[n]}) = p_1(t^{[n_1]}) p_2(t^{[n_2]}) = P_1(t) P_2(t) = P(t)$. ■

Let ω^* be the blossom of $\Omega \in \mathbb{P}_{n-1}$, that is, its blossom in $(n-1)$ variables. Let $P \in \mathbb{P}_n$ be defined by $P(t) := t\Omega(t)$, $t \in [0, 1]$. The blossom p of P (in n variables) is thus given by

$$p(t_1, \ldots, t_n) = \frac{1}{n} \sum_{i=1}^{n} t_i \omega^*(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n), \quad t_1, \ldots, t_n \in [0, 1].$$

As a special case, given distinct $a, b \in [0, 1]$, we obtain

$$p(a^{[n-i]}, b^{[i]}) = \frac{i}{n}\, b\, \omega^*(a^{[n-i]}, b^{[i-1]}) + \frac{n-i}{n}\, a\, \omega^*(a^{[n-1-i]}, b^{[i]}), \quad 1 \leq i \leq n-1.$$

Let us apply the latter formula with $a = 0$, $b = 1$. The Bézier points $p_0, \ldots, p_n$ of P are thus given by

$$p_0 = 0, \quad p_i = p(0^{[n-i]}, 1^{[i]}) = \frac{i}{n} \omega^*_{i-1} \quad \text{for } i = 1, \ldots, n. \tag{5.14}$$

Accordingly, as an element of the rational space $\widehat{\mathbb{P}}_n$, the Bézier points of the identity $X_1 = \frac{P}{\Omega}$ are given by (see (5.12))

$$\frac{p_0}{\omega_0} = 0, \quad \frac{p_i}{\omega_i} = \frac{i\omega^*_{i-1}}{i\omega^*_{i-1} + (n-i)\omega^*_i}, \quad 1 \leq i \leq n. \tag{5.15}$$

The positivity of ω_0^* and ω_{n-1}^* enables us to conclude that we always have

$$0 = \frac{p_0}{\omega_0} < \frac{p_1}{\omega_1}, \quad \frac{p_{n-1}}{\omega_{n-1}} < \frac{p_n}{\omega_n}.$$

We thus have to find conditions on $\omega_0^*, \ldots, \omega_{n-1}^*$ ensuring that

$$\frac{i\omega_{i-1}^*}{i\omega_{i-1}^* + (n-i)\omega_i^*} < \frac{(i+1)\omega_i^*}{(i+1)\omega_i^* + (n-i-1)\omega_{i+1}^*}, \quad 1 \le i \le n-2. \tag{5.16}$$

Assume that (5.16) is satisfied. Both denominators in (5.16) being positive –see (5.12)–, we can see that as soon as $\omega_{i-1}^* > 0$ for some integer i, $1 \le i \le n-1$, then ω_i^* is positive in turn. We already know that $\omega_0^*, \omega_{n-1}^*$ are positive. Hence, we now know that $\omega_0^*, \ldots, \omega_{n-1}^*$ are all positive. Moreover, again due to the positivity of the denominators, condition (5.16) can equivalently be written as follows:

$$i(n-i-1)\omega_{i-1}^*\omega_{i+1}^* < (i+1)(n-i){\omega_i^*}^2 \quad \text{for } i = 1, \ldots, n-2. \tag{5.17}$$

On the other hand, we know that the positivity of $\omega_0^*, \ldots, \omega_{n-1}^*$ guarantees that of $\omega_0, \ldots, \omega_n$ obtained via (5.12). Accordingly, we can summarise all previous discussion as follows.

th:reproP1 **Theorem 5.9.** *Given $\Omega = \sum_{i=0}^n \omega_i B_i^n \in \mathbb{P}_n$, the following properties are equivalent:*

(i) *the function Ω is positive on $[0,1]$ and (5.3) is a sequence of rational spaces which provides us with a sequence $\widehat{\mathbb{B}}_k$, $k \ge n$, of rational Bernstein operators reproducing $\mathbb{P}_1$;*

(ii) *$\omega_0, \ldots, \omega_n$ are given by (5.12), where $\omega_0^*, \ldots, \omega_{n-1}^*$ are any positive numbers satisfying*

$$i(n-i-1)\omega_{i-1}^*\omega_{i+1}^* < (i+1)(n-i){\omega_i^*}^2 \quad \textit{for } i = 1, \ldots, n-2.$$

One interesting advantage of reproduction of the identity is that it directly yields explicit expressions of the Bernstein operators. We can indeed complete Theorem 5.9 by the following one, which follows from (5.15) and (5.12).

h:reproP1bis **Theorem 5.10.** *Assume that $\Omega = \sum_{i=0}^n \omega_i B_i^n \in \mathbb{P}_n$, and that (ii) of Theorem 5.9 holds. Then, for each $k \ge n$ the Bernstein operator $\widehat{\mathbb{B}}_k$ based on the rational space $\widehat{\mathbb{P}}_k$ which reproduces the identity is given by*

$$\widehat{\mathbb{B}}_k F = \sum_{i=0}^k F\left(\frac{i\omega_{k-1,i-1}}{k\omega_{k,i}}\right)\widehat{B}_{k,i}, \quad F \in C^0([0,1]), \tag{5.18}$$

where, for each $k \ge n-1$, $\omega_{k,0}, \ldots, \omega_{k,k}$ are the (positive) Bézier points of Ω as an element of $\mathbb{P}_k$.

Remark 5.11. As observed in Remark 5.7, when $\mathbb{P}_1 \subset \widehat{\mathbb{P}}_n$ the space $\widehat{\mathbb{P}}_{n-1} := \frac{1}{\Omega}\mathbb{P}_{n-1}$ may fail to be a rational space of degree $(n-1)$. However, as soon as $\mathbb{P}_1$ is reproduced by a Bernstein operator based on $\widehat{\mathbb{P}}_n$, all $\omega_0^*, \ldots, \omega_{n-1}^*$ being positive (Theorem 5.9), $\widehat{\mathbb{P}}_{n-1}$ is indeed a rational space.

Remark 5.12. All suitable $\omega_0^*, \ldots, \omega_{n-1}^*$ ensuring condition (ii) of Theorem 5.9 can, for instance, be obtained by first choosing any positive ω_0^*, ω_1^*, and then successively selecting $\omega_2^*, \ldots, \omega_{n-1}^*$ such that

$$0 < \omega_i^* < \frac{i(n-i+1)}{(i-1)(n-i)} \frac{{\omega_{i-1}^*}^2}{\omega_{i-2}^*} \quad \text{for } i = 2, \ldots, n-1.$$

5.4. Comments on rational Bernstein operators

We would like to pay tribute to P. Piţul and P. Sablonnière for their interesting work [26] on a class of rational Bernstein operators, to which we will compare the previous subsection. As a special case of Theorem 5.4, we can say that the sequence of rational Bernstein operators defined by (5.18) satisfies

$$\lim_{k \to +\infty} \|\widehat{\mathbb{B}}_k F - F\|_\infty = 0 \text{ for all } F \in C^0([0,1]). \tag{5.19}$$

We would like to mention that this sequence is one instance of the sequences of rational positive operators studied in [26] (see also [29]). For each integer k, the authors of [26] considered a positive operator $\mathcal{R}_k$

$$\mathcal{R}_k F = \frac{\sum_{i=0}^k \overline{w}_{k,i} F(t_{k,i}) B_i^k}{\Omega_k}, \quad F \in C^0([0,1]), \quad \text{with } \Omega_k := \sum_{i=0}^{k-1} w_{k,i} B_i^{k-1}, \tag{5.20}$$

the $w_{k,i}$, $0 \le i \le k$, being any positive numbers chosen so as to satisfy

$$\frac{w_{k,i-1} w_{k,i+1}}{{w_{k,i}}^2} < \left(\frac{i+1}{i}\right)\left(\frac{k-i}{k-i-1}\right), \quad 1 \le i \le k-2. \tag{5.21}$$

Moreover in the numerator, the $w_{k,i}$, $t_{k,i}$, are given by

$$\overline{w}_{k,i} = w_{k,0}, \quad \overline{w}_{k,k} = w_{k,k-1}, \quad \overline{w}_{k,i} = \frac{i}{k} w_{k,i-1} + \left(1 - \frac{i}{k}\right) w_{k,i}, \tag{5.22}$$

and

$$t_{k,0} := 0, \quad t_{k,k} := 1, \quad t_{k,i} = \frac{i w_{k,i-1}}{i w_{k,i-1} + (k-i) w_{k,i}}, \tag{5.23}$$

That Ω_k belongs to $\mathbb{P}_{k-1}$ and has positive Bézier points is required to guarantee that the space $\frac{1}{\Omega_k}\mathbb{P}_k$ is a rational space of degree k with linear precision. All other requirements (5.21), (5.22), (5.23), are intended to ensure reproduction of $\mathbb{P}_1$ by the operator $\mathcal{R}_k$, along with strictly increasing $t_{k,i}$, $0 \le i \le k$. All these conditions are satisfied by our operators (5.18). The operators $\mathcal{R}_k$ form an infinite sequence of Bernstein operators in the Chebyshevian sense. However, while in our case $\Omega_k = \Omega \in \mathbb{P}_{n-1}$ for all $k \ge n$, in [26] we have a sequence of Bernstein operators respectively based on the rational spaces $\frac{1}{\Omega_k}\mathbb{P}_k$, each Ω_k being a priori chosen independently of the others. Nevertheless, in the more general situation addressed in [26], the main result of uniform convergence of $\mathcal{R}_kF$ to F for any continuous function F ([26], Theorem 7.1), is obtained under the additional requirement that there exists a function $\varphi \in C^0([0,1])$ such that

$$\Omega_k = B^*_{k-1}\varphi \quad \text{for all } k. \tag{5.24}$$

Clearly, Theorems 5.9 and 5.10 are not within this context, unless our Ω is reproduced by $\mathbb{B}^*_{k-1}$ for all $k \ge n$, that is unless Ω belong to $\mathbb{P}_1$. If so, (5.18) can be rewritten with $\omega_{k,i} = \Omega(\frac{i}{k})$.

Assume that we are not in the latter special case. Using the notations introduced in Remark 2.4, we can always state that

$$\Omega = B^*_{k-1}\varphi_k, \quad \text{with } \varphi_k := L_{k-1}\Omega, \quad k \ge n-1.$$

We know that the sequence φ_k, $k \ge n$, is uniformly convergent to Ω on $[0,1]$. Accordingly, in the special situation addressed in Theorems 5.9 and 5.10, the convergence result of Theorem 5.4 illustrates Remark 7.2 of [26]. Indeed, it is mentioned there that condition (5.24) can be weakened and replaced by $\Omega_k = B^*_{k-1}\varphi_k$ where φ_k is a sequence of continuous functions uniformly convergent to some function φ on $[0,1]$.

To conclude this section, let us mention that, prior to [26], rational operators of the Bernstein type were generally defined as

$$\mathcal{Q}_kF := \frac{\sum_{i=0}^k w_{k,i}F\left(\frac{i}{k}\right)B_i^k}{\sum_{i=0}^k w_{k,i}B_i^k}, \quad F \in C^0([0,1]), \tag{5.25}$$

where the $w_{k,i}$, $i = 0,\dots,k$ are any positive numbers [7, 9, 25]. This is indeed an easy way to obtain uniform rational approximation of continuous functions as a consequence of Korovkin's theorem and (2.3). However, in general such operators are not Bernstein operators in the Chebyshevian sense. We would like to draw the reader's attention on the fact that, even if the $w_{k,i}$'s are chosen so as to have linear precision in the corresponding rational space, the operator $\mathcal{Q}_k$ defined in (5.25) cannot reproduce $\mathbb{P}_1$ unless it is the classical Bernstein operator $\mathbb{B}^*_k$, that is, if all $w_{k,i}$'s are equal.

§6. Final remarks

It can be advantageous to regard polynomial spaces on closed bounded intervals as instances of EC-spaces, and thereby to replace the simple ordinary derivatives by generalised ones, more difficult to handle. We hope that we have convinced the reader of this fact via the question of Bernstein operators. Further issues concerning approximation by polynomial or rational Bernstein operators could be of interest. Below we cite only a few possible ones:

- It may be worthwhile looking deeper into the question of polynomial (rational) approximation with shape parameters.

- As in [26] we have considered reproduction of the identity by rational Bernstein operators. Via Lemma 5.8, one could address the more general problem: $U \in \mathbb{P}_n$ being given, how to determine Ω so that the corresponding rational space $\widehat{\mathbb{P}}_n$ will provide us with a Bernstein operator reproducing U?

- Is it possible to build polynomial or rational Bernstein operators with prescribed $\zeta_0, \dots, \zeta_n$?

- In connection with the previous question, can we choose positive $w_{k,i}$, $i = 0, \dots, k$, so that the operator $\mathcal{Q}_k$ defined in (5.25) will be a Bernstein operator?

Finally we would like to conclude this article by mentioning that similar results do exist for polynomial and rational Schoenberg operators. In that case, they are obtained when considering polynomial splines as examples of *piecewise Chebyshevian splines* (*i.e.*, splines with pieces taken from different EC-spaces) which are *good for design* (in the sense that they possess refinable B-spline bases, or blossoms as well) [21].

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